STUDIES IN COMPUTATIONAL MATHEMATICS 12
editors: C.K. CHUI, P. MONK and L. WUYTACK

TOPICS IN MULTIVARIATE APPROXIMATION AND INTERPOLATION

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PREFACE

Multivariate Approximation and Interpolation has been an active research area in applied mathematics, for many years, and has had impact on various applications, in computer aided geometric design, in mathematical modeling, in computations with large scale data, in signal analysis and image processing, to mention a few. More recently, approximation theoretical ideas have shown to be useful even in the analysis of learning algorithms. It is the purpose of this book to give an overview of some - although selective - areas in this field in order to have a compact and up-to-date edition of issues in this stimulating topic. We hope that such a volume will be a good basis for graduate students and young researchers to dive into the subject, and a valuable resource of information for all researchers working in the field.

The eleven articles in this book are written by leading experts, who have been invited to communicate their experience and knowledge in a particular subject. The contributions are mainly written as surveys, with much background to start with, with a presentation of the main achievements from the past to the present, leading the reader finally to the forefront of research. The authors were also asked to provide an appropriate, although not comprehensive, list of references. We thank all the contributors for their support in this ambitious project, and for their immense efforts to make this collection of articles a highly valuable piece of work.

A short description to each chapter follows:

Durrmeyer Operators and Their Natural Quasi-Interpolants deals with a class of new polynomial reproducing quasi interpolants on simplices which were recently discovered by two of the authors. Their construction deviates from the usual approach using summability, and is based on new identities for Bernstein basis polynomials. The article not only provides a survey on the spectral analysis and the approximation properties of these operators, but in addition points to an interesting connection with hypergeometric series. In particular, a striking result on the property of a certain kernel function being pointwise completely monotonic is proved. The results are expected to provide a useful alternative for the construction of high order linear approximation schemes in function spaces of several variables.

The second chapter *Three Families of Nonlinear Subdivision Schemes* is written by Nira Dyn, who has been at the forefront of research in subdivision, for many years. The present article describes three more recent issues in the field, which deal with nonlinear schemes. First, control polygons with strong nonuniformity concerning the length of edges, are discussed. Next, local weighted essentially non oscillatory schemes are constructed which have the advantage to depend continuously on the data. And finally, subdivision schemes on manifolds are derived which are modifications of converging linear schemes, and which are analysed by their proximity to these.

The chapter *Parameterization for Curve Interpolation* by M. Floater and T. Surazhsky considers the approximation order for curve interpolation by parametric spline curves. The authors explain that, for the clamped cubic spline interpolant, the chord length parameterization gives full order of approximation as measured in the Hausdorff distance. Moreover, a bootstrapping method for improving the parameterization is proposed in order to obtain optimal approximation order for higher degree spline interpolants, such as the two-point quintic Hermite scheme of order 6. A short survey of degree-reduced schemes is also included.

In the chapter *Refinable Multivariate Spline Functions*, T. Goodman and D. Hardin present a very general view on what is probably the most important building block in wavelet analysis: refinable functions and especially those from spline spaces in one and more dimensions. Both gridded data and general triangulations are considered. With the former, the well-known box-splines and the so-called new multi-box-splines are linked. The latter are addressed in connection with continuous differentiable spline functions and with piecewise linear splines. The article is a very comprehensive review with several examples, where the numerical stability of the functions in the presented approaches is of special interest.

In the chapter *Adaptive Wavelets for Sparse Representations of Scattered Data* A. Kunoth considers the problem of scattered data fitting by a sparse wavelet representation. The presented schemes are based on least-squares approximation and wavelet thresholding. The considered methods are data-dependent and operate by adaptive refinement in a coarse-to-fine manner. In particular, the initial step of typical wavelet methods is avoided, where gridded data on a "finest" resolution level must be generated. The chapter also discusses the main ideas for solving large scattered data problems including the multilevel regularisation and the treatment of outliers in a concise way. With this chapter the author gives a very good survey on recent developments in this area.

The author of the chapter *Ready-to-Blossom Bases in Chebyshev Spaces* is a well-known expert especially in the theory of blossoming. In the present review paper, M.-L. Mazure gives a comprehensive survey on the concept of blossoming and the fundamental notion of extended Chebychev spaces. For the latter, characterisations are presented in many equivalent formulations, some of them known and reviewed here, some of them new. For the former, both existence and their properties are discussed and, for instance, the relationship between blossoms and Bernstein bases and the existence of Bézier points is explained. And, of course, special attention is given to blossoms in the EC (extended Chebyshev) spaces.

A comprehensive survey along with some new results on the structural analysis of subdivision surfaces near extraordinary vertices is offered in the chapter *Structural Analysis of Subdivision Surfaces – A Summary* by J. Peters and U. Reif. For "standard" surface subdivision schemes, whose subdivision matrix has a double subdominant eigenvalue, the issues of normal and C^1-continuity are discussed in detail. Here, the authors extend the known results to cases where the generating functions of the scheme may be linearly dependent. Moreover, a simplified test of injectivity for the so-called characteristic map is developed for subdivision schemes with certain symmetry properties. The Doo-Sabin scheme serves as an illustration of these new techniques. The C^2-regularity and corresponding constraints for the subdivision matrix are also discussed. The chapter closes with a detailed analysis of the limit curvature at extraordinary vertices, which is very useful for understanding the visual artifacts in specific subdivision surfaces.

Polynomial Interpolation in Several Variables: Lattices, Differences, and Ideals. In this chapter T. Sauer points out that when passing from one to several variables, the nature and structure of polynomial interpolation changes completely. The solvability of an interpolation problem with respect to a given finite dimensional space of multivariate polynomials does not only depend on the number of the nodes but significantly on their geometric position. This makes the theory of interpolation in several variables a highly difficult and non-trivial problem. The main reason is the loss of the Haar condition in domains different from univariate intervals or S^1. The author gives an excellent survey of some basic constructions of interpolation lattices which emerge from the geometric characterization due to Chung and Yao. Depending on the structure of the specific interpolation problem, there are different representations of the interpolation polynomial and of the error formulas, reflecting the underlying point geometry. In addition, the close relationship with algebraic concepts such as constructive ideal theory is pointed out.

A particularly elegant way of solving multivariate interpolation and approximation problems is provided by kernels or radial basis functions. These have plenty of applications, since they provide meshless methods for solving partial differential equations and are in the core of modern techniques for machine learning. The chapter *Computational Aspects of Radial Basis Function Approximation* by H. Wendland surveys recent progress in numerical methods connected to kernel techniques. Reduction of problem complexity and improvement of stability are the most important computational bottlenecks in this area. Both are treated comprehensively, in particular by multipole expansions, domain decompositions, partitions of unity, multilevel techniques, and regularization by smoothing.

Kernels and regularization are the link to the paper *Learning Theory: From Regression to Classification* by Q. Wu, Y. Ying, and D. X. Zhou which looks at recent developments in machine learning from the viewpoint of approximation theory. In particular, a regularization approach in reproducing kernel Hilbert spaces is used to analyze errors of regression and classification algorithms. This field connects multivariate approximation to optimization and stochastic processes, and it has quite a promising future due to its importance for the design of intelligent systems in engineering.

The final chapter *Coherent States from Nonunitary Representations* by G. Zimmermann provides an interesting and powerful alternative to periodic wavelets on the unit circle by employing Möbius transformations as generators for the building blocks in the analysis and in the synthesis process. The usual unitary representations of this group of transformations being not square integrable, however, the usual "wavelet" construction has to be modified. It is now built on a nonunitary representation and its contragredient counterpart. The chapter also deals with these aspects in a general, and abstract, way in order to provide the essential ingredients for this extension of wavelet-type expansion of functions in appropriate function spaces.

Many people have contributed to the production of the book. All the articles are peer-refereed and carefully edited. Our thanks go to the referees for their valuable support, guaranteeing top scientific standard of all chapters. During the editing procedure, we got help from Dr. Elena Berdysheva and Dr. Georg Zimmermann to whom we are grateful, in particular, for compiling the index. Last not least, we would like to thank the series editors, and the publisher for their kind cooperation.

Martin D. Buhmann
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The editors

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Topics in Multivariate Approximation and Interpolation
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Durrmeyer Operators and Their Natural Quasi-Interpolants

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Dedicated to Professor Charles K. Chui on the occasion of his 65th birthday.

Abstract

This paper provides a survey on spectral analysis and approximation order of our quasi-interpolants of Durrmeyer type on simplices, together with various new aspects and achievements. The latter include Bernstein type inequalities which are proved using a striking property of appropriately modified Durrmeyer operators, namely, their kernel functions are pointwise completely monotonic.

Key words: Bernstein basis polynomial, Bernstein inequality, completely monotonic sequence, Durrmeyer operator, hypergeometric series, Jackson-Favard estimate, Jacobi polynomial, K-functional, Laplace type integral, Legendre differential operator, positive operator, quasi-interpolants, Voronovskaja theorem

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1. Introduction

The construction of quasi-interpolant operators through linear combinations of (Bernstein-)Durrmeyer operators has a long history in Approximation Theory. Durrmeyer operators have several desirable properties such as positivity and stability, and their analysis can be performed using their elegant spectral properties. Their approximation order is low, however, and for this reason quasi-interpolants with better approximation properties are necessary for more efficient approximation. In our aim at constructing good quasi-interpolants on triangulated domains, the natural first step is to consider a single triangle - or a simplex in higher dimensions.

A comprehensive description of our previous results in this direction is included as part of this article. We emphasize the close relation of quasi-interpolants to certain partial differential operators on the simplex, which are generalizations of the Legendre differential operator and its Jacobi-type analogue

$$P_r^{\alpha,\beta} := w_{\alpha,\beta}^{-1}(x)\frac{d^r}{dx^r}\left[w_{\alpha,\beta}(x)x^r(1-x)^r\frac{d^r}{dx^r}\right],$$

where $w_{\alpha,\beta}(x) = x^\alpha(1-x)^\beta$ and $\alpha, \beta > -1$ define the Jacobi weight for the standard interval $[0, 1]$. In addition to this survey we also present new results which lead to a Bernstein estimate for the aforementioned differential operators (Section 6) and to direct estimates of the error of approximation of our quasi-interpolants by newly defined K-functionals on the simplex (Section 7). The key result in order to prove the Bernstein inequality is a beautiful property of the sequence of appropriately modified Durrmeyer operators: their kernels constitute a pointwise completely monotonic sequence (Theorem 2). Here we employ methods of Koornwinder and Askey for the Laplace integral of Jacobi-polynomials and the characterization of completely monotonic sequences by Hausdorff's theorem (Section 4).

The structure of the paper is as follows. In Section 2 we give the definition of the Durrmeyer operators (with Jacobi weights), and in Section 3 we review their spectral properties, see Theorem 1. Section 4 deals with the kernel function of the appropriately modified Durrmeyer operator, and provides the striking result of Theorem 2 showing the pointwise complete monotonicity of the associated kernels. We then give the definition of our quasi-interpolants in Section 5, together with the adequate partial differential operators of Jacobi type. Their spectral analysis leads to a valuable representation of the quasi-interpolants as a linear combination of Durrmeyer operators, in Theorem 9. Section 6 is devoted to the proof of the Bernstein inequalities, which are stated in Theorem 11 and Theorem 12. This is the second key section of the paper, which contains new and unpublished material. Its application in Section 7 follows along the lines of classical Approximation Theory and provides a valuable and elegant extension of several properties of the Durrmeyer operator to our quasi-interpolants: The estimate of Jackson-Favard type, the Voronovskaja type theorem (including its 'strong' version) and the so-called direct estimate in terms of the proper K-functional. Rather than giving complete references for each result within the text, we conclude in Section 8 with historical remarks in order to point out the development of the main results.

2. The Bernstein Basis Functions

The standard simplex in $\mathbb{R}^d$ is given by

$$\mathbf{S}^d := \{ (x_1, \ldots, x_d) \in \mathbb{R}^d \mid 0 \le x_1, \ldots, x_d \le 1 \, , \; x_1 + \cdots + x_d \le 1 \} \, .$$

We shall use barycentric coordinates

$$\mathbf{x} = (x_0, x_1, \ldots, x_d) \, , \qquad x_0 := 1 - x_1 - \ldots - x_d \, ,$$

in order to define the d-variate Bernstein basis polynomials. Namely, for given $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^{d+1}$

$$B_\alpha(x_1, \ldots, x_d) := \binom{|\alpha|}{\alpha} \mathbf{x}^\alpha := \frac{|\alpha|!}{\alpha_0! \alpha_1! \cdots \alpha_d!} \, x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_d^{\alpha_d} \, .$$

Here, we use standard multi-index notation. We also allow $\alpha \in \mathbb{Z}^d$ with $|\alpha| := \alpha_0 + \cdots + \alpha_d \in \mathbb{N}$. It is then convenient to put $B_\alpha \equiv 0$ if one α_i is negative.

For given $n \in \mathbb{N}$, the Bernstein basis polynomials $\{B_\alpha \mid \alpha \in \mathbb{N}_0^{d+1} \, , \; |\alpha| = n\}$ are a basis for $\mathbf{P}_n = \mathbf{P}_n^d$, the space of d-variate algebraic polynomials of (total) degree n. They are used for the definition of various polynomial operators. In this paper, we study quasi-interpolants based on the Bernstein-Durrmeyer operators with Jacobi weights. Here, the weight function is given by

$$\omega_\mu(x_1, \ldots, x_d) := \mathbf{x}^\mu = x_0^{\mu_0} x_1^{\mu_1} \cdots x_d^{\mu_d} \, ,$$

where $\mu = (\mu_0, \mu_1, \ldots, \mu_d) \in \mathbb{R}^{d+1}$ with $\mu_i > -1$, $i = 0, \ldots, d$. Whence, $|\mu| := \mu_0 + \mu_1 + \cdots + \mu_d > -d - 1$.

On the simplex, we use the (weighted) inner product

$$\langle f | g \rangle_\mu := \int_{\mathbf{S}^d} \omega_\mu \, f \, g \tag{1}$$

to define the (Bernstein-)Durrmeyer operator of degree n,

$$\mathbf{M}_{n,\mu} \, : \, f \mapsto \mathbf{M}_{n,\mu}(f) := \sum_{|\alpha|=n} \frac{\langle f | B_\alpha \rangle_\mu}{\langle \mathbf{1} | B_\alpha \rangle_\mu} \, B_\alpha \, . \tag{2}$$

Here, $\mathbf{1}$ denotes the function constant equal one. It is well-known that, for $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^{d+1}$, with $|\alpha| = n$,

$$\begin{aligned} \langle \mathbf{1} | B_\alpha \rangle_\mu &= \binom{|\alpha|}{\alpha} \frac{\Gamma(\alpha_0 + \mu_0 + 1)\Gamma(\alpha_1 + \mu_1 + 1) \cdots \Gamma(\alpha_d + \mu_d + 1)}{\Gamma(|\alpha| + |\mu| + d + 1)} \\ &= \frac{\Gamma(n+1)}{\Gamma(n + |\mu| + d + 1)} \prod_{i=0}^{d} \frac{\Gamma(\alpha_i + \mu_i + 1)}{\Gamma(\alpha_i + 1)} \, . \end{aligned} \tag{3}$$

For the unweighted case (where $\mu = (0,0,\ldots,0)$) this recovers the formula

$$\int_{\mathbf{S}^d} B_\alpha = \frac{n!}{(n+d)!}\,, \quad \alpha \in \mathbb{N}_0^{d+1}\,,\ |\alpha| = n\,.$$

3. Spectral Properties of the Durrmeyer Operators

The Durrmeyer operator is usually considered on the domain $L^p_\mu(\mathbf{S}^d)$, $1 \le p < \infty$, which is the weighted L^p space consisting of all measurable functions on S^d such that

$$\|f\|_{p,\mu} := \left(\int_{\mathbf{S}^d} \omega_\mu\, |f|^p\right)^{1/p}$$

is finite. For $p = \infty$, the space $C(\mathbf{S}^d)$ of continuous functions is considered instead. In this setting the following properties are more or less obvious.

- *The operator is positive:* $\mathbf{M}_{n,\mu}(f) \ge 0$ for every $f \ge 0$.
- *It reproduces constant functions:* $\mathbf{M}_{n,\mu}(p) = p$ for $p \in \mathbf{P}_0$.
- *It is contractive:* $\|\mathbf{M}_{n,\mu}(f)\|_{p,\mu} \le \|f\|_{p,\mu}$ for every $f \in L^p_\mu(\mathbf{S}^d)$.

The most striking and useful property refers to the Hilbert space setting,

$$\mathbf{H} := L^2_\mu(\mathbf{S}^d)\,.$$

This space can be written as the sum of spaces of orthogonal polynomials,

$$L^2_\mu(\mathbf{S}^d) = \sum_{m=0}^{\infty} \mathbf{E}_{m,\mu}\,, \quad \text{where}$$

$$\mathbf{E}_{0,\mu} := \mathbf{P}_0 \quad \text{and} \quad \mathbf{E}_{m,\mu} := \mathbf{P}_m \cap \mathbf{P}_{m-1}^{\perp} \quad \text{for} \quad m > 0\,.$$

Here, orthogonality refers to the weighted inner product (1). It is clear that $\mathbf{M}_{n,\mu}$ is a bounded self-adjoint operator on $\mathbf{H}$. Its spectrum is given by the following result.

Theorem 1. *For all $n \in \mathbb{N}$, the spaces $\mathbf{E}_{m,\mu}$, $m \ge 0$, are eigenspaces of the Durrmeyer operator, and*

$$\mathbf{M}_{n,\mu}(p_m) = \gamma_{n,m,\mu}\, p_m \quad \textit{for} \quad p_m \in \mathbf{E}_{m,\mu}\,,$$

where

$$\gamma_{n,m,\mu} := \frac{n!}{(n-m)!}\,\frac{\Gamma(n+d+|\mu|+1)}{\Gamma(n+d+|\mu|+m+1)}\,, \quad \textit{for} \quad n \ge m\,,$$

while

$$\gamma_{n,m,\mu} = 0 \quad \textit{for} \quad n < m\,.$$

In particular,

$$\gamma_{n,0,\mu} = 1\,, \quad 0 \le \gamma_{n,m,\mu} < 1\,, \quad m > 0\,,$$

and

$$\lim_{n\to\infty} \gamma_{n,m,\mu} = 1 \quad \textit{for fixed } m\,. \tag{4}$$

Hence, for $f = \sum_{m=0}^{\infty} p_m$, with $p_m \in \mathbf{E}_{m,\mu}$, we find $\mathbf{M}_{n,\mu}(f) = \sum_{m=0}^{n} \gamma_{n,m,\mu}\, p_m$. In particular, the restrictions $\mathbf{M}_{n,\mu}|_{\mathbf{P}_k}$ act as isomorphisms on the spaces $\mathbf{P}_k$ as long as $k \le n$.

4. The Kernel Function

According to equations (2) and (3), the Durrmeyer operator of degree n can be written as

$$\{\mathbf{M}_{n,\mu}(f)\}(\mathbf{y}) = \int_{\mathbf{S}^d} \omega_\mu(\mathbf{x})\, f(\mathbf{x})\, K_{n,\mu}(\mathbf{x},\mathbf{y})\, d\mathbf{x}$$

with its kernel given by

$$K_{n,\mu}(\mathbf{x},\mathbf{y}) := \frac{\Gamma(n+|\mu|+d+1)}{\Gamma(n+1)} \sum_{|\alpha|=n} \left(\prod_{i=0}^{d} \frac{\Gamma(\alpha_i+1)}{\Gamma(\alpha_i+\mu_i+1)}\right) B_\alpha(\mathbf{x})\, B_\alpha(\mathbf{y}) \,.$$

Putting

$$\underline{\mu} := \min_i \mu_i \,,$$

we are going to study properties of the modified kernel

$$\begin{aligned} T_{n,\mu}(\mathbf{x},\mathbf{y}) :&= \frac{\Gamma(n+\underline{\mu}+1)}{\Gamma(n+|\mu|+d+1)}\, K_{n,\mu}(\mathbf{x},\mathbf{y}) \\ &= \sum_{|\alpha|=n} \frac{\Gamma(n+\underline{\mu}+1)}{\prod_{i=0}^{d}\Gamma(\alpha_i+\mu_i+1)} \binom{n}{\alpha} \mathbf{x}^\alpha\, \mathbf{y}^\alpha \,, \qquad \mathbf{x},\mathbf{y} \in \mathbf{S}^d. \end{aligned} \tag{5}$$

This kernel is non-negative, but we are going to prove much more. Namely, under a slight restriction on the exponents of the Jacobi weight, the forward differences of the sequence $(T_{n,\mu}(\mathbf{x},\mathbf{y}))_{n\ge 0}$ alternate in sign. The result seems to be new even in the univariate setting.

Theorem 2. *Let $\mu = (\mu_0, \mu_1, \dots, \mu_d)$ be such that $\mu_i \ge -1/2$, $i = 0, \dots, d$. Then, for every $\mathbf{x},\mathbf{y} \in \mathbf{S}^d$, the sequence $(T_{n,\mu}(\mathbf{x},\mathbf{y}))_{n\ge 0}$ is bounded and completely monotonic; i.e., the inequalities*

$$T^{(r)}_{n,\mu}(\mathbf{x},\mathbf{y}) := (-1)^r \Delta^r T_{n,\mu}(\mathbf{x},\mathbf{y}) = \sum_{\ell=0}^{r} (-1)^\ell \binom{r}{\ell} T_{n+\ell,\mu}(\mathbf{x},\mathbf{y}) \ge 0$$

hold true for all $r, n \ge 0$.

Here, $(\Delta\nu_n)_{n\ge 0} = (\nu_{n+1} - \nu_n)_{n\ge 0}$ denotes the forward difference of the sequence $(\nu_n)_{n\ge 0}$. For the proof of this result, we make use of the characterization of completely monotonic sequences due to Hausdorff.

Lemma 3. *(see [28], Chapter III, Theorem 4a) A real sequence $(\nu_n)_{n\ge 0}$ is completely monotonic if and only if there exists a non-decreasing bounded function g on $[0,1]$ such that*

$$\nu_n = \int_0^1 t^n \, dg(t), \qquad n \in \mathbb{N}_0 .$$

Here, the integral is to be understood as a Lebesgue-Stieltjes integral.

Remarks. The following facts will be useful for our discussion of complete monotonicity.

(a) *The sequences $(q^n)_{n\geq 0}$, with $0 \leq q \leq 1$, are completely monotonic.* This result is obvious.

(b) *The sum and the product of two completely monotonic sequences are completely monotonic.* The first statement is again trivial, and the second statement follows from

$$-\Delta(c_n d_n) = (-\Delta c_n)d_n + c_{n+1}(-\Delta d_n)$$

by induction.

(c) *For given $\mu_0 \geq \mu_1 > -1$ the sequence*

$$c_n = \frac{\Gamma(n+\mu_1+1)}{\Gamma(n+\mu_0+1)}, \quad n \geq 0 ,$$

is completely monotonic. This follows from the formula

$$(-1)^k \Delta^k c_n = \frac{(\mu_0-\mu_1)_k}{(n+1+\mu_0)_k}\, c_n , \quad k,n \geq 0 ,$$

with $(a)_0 := 1$ and $(a)_k := a(a+1)\cdots(a+k-1)$ for $k > 0$, the so-called shifted factorial or Pochhammer symbol.

(d) *The sequence of integrals $c_n = \int f_n \, dm$, $n \geq 0$, of a pointwise completely monotonic family $(f_n)_{n\geq 0}$ of functions which are integrable with respect to the non-negative measure dm, is completely monotonic.*

For the proof of Theorem 2 it is sufficient to consider the case

$$\mu_0 \geq \mu_1 \geq \cdots \geq \mu_d \geq -1/2 , \quad \text{whence } \underline{\mu} = \mu_d ,$$

since the kernel is invariant modulo a permutation of the variables. Under this assumption, we use induction on d, the number of variables. For $d = 1$, the statement is the special case $t = 1$, $(\alpha, \beta) = (\mu_0, \mu_1)$ of the following result.

Lemma 4. *Let $\alpha \geq \beta \geq -1/2$. Then, for every $x, y, t \in [0,1]$, the sequence*

$$\begin{aligned}\nu_n(x,y;t) &= \nu_n^{(\alpha,\beta)}(x,y;t) \\ &= \sum_{k=0}^{n} \frac{\Gamma(n+\beta+1)}{\Gamma(k+\alpha+1)\Gamma(n-k+\beta+1)} \binom{n}{k} (xy)^k \big((1-x)(1-y)t\big)^{n-k} ,\end{aligned}$$

$n \geq 0$, is bounded and completely monotonic.

Remark. The restriction on the Jacobi exponents is crucial. For example, a simple calculation shows that

$$-\Delta\nu_1^{(\beta,\beta)}\left(\frac{1}{2},\frac{1}{2};1\right) = \frac{1}{\Gamma(\beta+1)}\,\frac{2\beta+1}{8(\beta+1)}\,,$$

which is negative for $-\frac{1}{2} > \beta > -1$.

Taken the result of Lemma 4 for granted, the *proof of Theorem 2* is finished by the following induction step.

For $d \geq 2$, we write the kernel $T_{n,\mu} = T^d_{n,\mu}$ in equation (5) in terms of kernels of fewer variables (For clarity, we mark the number of variables as a superscript). We put $\mathbf{x} = (x_0, \mathbf{x}^*)$ with $\mathbf{x}^* = (x_1, \ldots, x_d)$ and $x_0 = (1 - x_1 - \cdots - x_d)$, and $\mathbf{y} = (y_0, \mathbf{y}^*)$ with analogous notation. Also, $\alpha = (\alpha_0, \alpha^*)$ with $\alpha^* = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, and $\mu = (\mu_0, \mu^*)$ with $\mu^* = (\mu_1, \ldots, \mu_d) \in \mathbb{R}^d$.

If $x_0 = 1$, then $x_1 = \cdots = x_d = 0$ and

$$T^d_{n,\mu}(\mathbf{x},\mathbf{y}) = \frac{1}{\prod_{i=1}^d \Gamma(\mu_i+1)}\,\frac{\Gamma(n+\mu_d+1)}{\Gamma(n+\mu_0+1)}\,y_0^n\,, \qquad n \geq 0\,,$$

which is completely monotonic by an application of cases (a)-(c) of the remarks above. The same argument applies if $y_0 = 1$. So we may assume henceforth that

$$0 \leq x_0, y_0 < 1\,.$$

Here, both $\tilde{\mathbf{x}} := \frac{1}{1-x_0}\mathbf{x}^*$ and $\tilde{\mathbf{y}} := \frac{1}{1-y_0}\mathbf{y}^*$ are elements of $\mathbf{S}^{d-1}$. By simple computations and by letting $k := \alpha_0$, we obtain

$$\begin{aligned} T^d_{n,\mu}(\mathbf{x},\mathbf{y}) &= \sum_{k=0}^{n} \frac{\Gamma(n+\mu_d+1)}{\Gamma(k+\mu_0+1)}\binom{n}{k}(x_0y_0)^k\left((1-x_0)(1-y_0)\right)^{n-k} \\ &\qquad \sum_{|\alpha^*|=n-k} \frac{1}{\prod_{i=1}^d \Gamma(\alpha_i+\mu_i+1)}\binom{n-k}{\alpha^*}(\tilde{\mathbf{x}})^{\alpha^*}(\tilde{\mathbf{y}})^{\alpha^*} \\ &= \sum_{k=0}^{n} a_{k,n}(x_0,y_0)\,T^{d-1}_{n-k,\mu^*}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\,, \end{aligned} \tag{6}$$

with

$$a_{k,n}(x_0,y_0) := \frac{\Gamma(n+\mu_d+1)}{\Gamma(k+\mu_0+1)\Gamma(n-k+\mu_d+1)}\binom{n}{k}(x_0y_0)^k\left((1-x_0)(1-y_0)\right)^{n-k}.$$

Using the induction hypothesis we find - according to Lemma 3 - for each pair $(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ a bounded, nondecreasing function g^* such that

$$\nu_n^* := T^{d-1}_{n,\mu^*}(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) = \int_0^1 t^n\,dg^*(t)\,, \qquad n \geq 0\,.$$

Inserting this in equation (6),

$$T^d_{n,\mu}(\mathbf{x},\mathbf{y}) = \int_0^1 \sum_{k=0}^{n} a_{k,n}(x_0,y_0)\,t^{n-k}\,dg^*(t)\,.$$

Here, the integrand is given by the sequence $\nu_n(x_0, y_0; t)$ considered in Lemma 4, by putting $(\alpha, \beta) = (\mu_0, \mu_d)$, and the induction step is completed by applying item (d) of the remarks above. This finishes the proof of Theorem 2.

For the *proof of Lemma 4*, it is sufficient to assume $\alpha > \beta > -\frac{1}{2}$, since the limit case $\alpha = \beta$ or $\beta = -\frac{1}{2}$ then follows by continuity. Here we make use of Koornwinder's integral representation (of Laplace type) for the normalized Jacobi polynomials which are given by a hypergeometric series as follows:

$$\begin{aligned} R_n^{(\alpha,\beta)}(x) := \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)} &:= F_1^2\big(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\big) \\ &= \big(\frac{x+1}{2}\big)^n \, F_1^2\big(-n, -n-\beta; \alpha+1; \frac{x-1}{x+1}\big) \end{aligned} \tag{7}$$

for $\alpha, \beta > -1$. The latter identity follows from Euler's linear transformation,

$$F_1^2(a, b; c; z) = (1-z)^{-a} \, F_1^2\big(a, c-b; c; \frac{z}{z-1}\big) \,,$$

(cf. [22], Section 2.4). Koornwinder's result (see [21], Section 3) reads as follows; for an easy analytic proof we refer to Askey [2].

Lemma 5. *For $\alpha > \beta > -\frac{1}{2}$ we have*

$$\begin{aligned} &R_n^{(\alpha,\beta)}(x) \\ &= \frac{2\Gamma(\alpha+1)}{\Gamma(\beta+\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(\alpha-\beta)} \int_{u=0}^{1} \int_{\varphi=0}^{\pi} \Big[\frac{x+1}{2} + \frac{x-1}{2}u^2 + u\sqrt{x^2-1}\cos\varphi\Big]^n \\ &\qquad\qquad u^{2\beta+1} \, (1-u^2)^{\alpha-\beta-1} \, \big(\sin\varphi\big)^{2\beta} \, d\varphi \, du \,. \end{aligned}$$

With this result at hand, the proof is finished by a straightforward, but lengthy computation. Using (7), we find

$$\begin{aligned} \nu_n(x, y; t) &= \frac{[(1-x)(1-y)t]^n}{\Gamma(\alpha+1)} F_1^2\big(-n, -n-\beta; \alpha+1; \frac{xy}{(1-x)(1-y)t}\big) \\ &= \frac{1}{\Gamma(\alpha+1)} \big[(1-x)(1-y)t - xy\big]^n \, R_n^{(\alpha,\beta)}\Big(\frac{(1-x)(1-y)t + xy}{(1-x)(1-y)t - xy}\Big) \,, \end{aligned}$$

and Koornwinder's integral gives

$$\nu_n(x, y; t) = \int_{u=0}^{1} \int_{\varphi=0}^{\pi} \big[\Phi(x, y, t; u, \varphi)\big]^n \, dm_{\alpha,\beta}(u, \varphi) \,. \tag{8}$$

Here,

$$\Phi(x, y, t; u, \varphi) = (1-x)(1-y)t + xyu^2 + 2\varepsilon\sqrt{(1-x)(1-y)txy} \, u \, \cos\varphi \tag{9}$$

with $\varepsilon = \mathrm{sign}((1-x)(1-y)t - xy)$ and

$$dm_{\alpha,\beta}(u, \varphi) = \frac{2}{\Gamma(\beta+\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(\alpha-\beta)} \, u^{2\beta+1} \, (1-u^2)^{\alpha-\beta-1} \, \big(\sin\varphi\big)^{2\beta} \, d\varphi \, du \,,$$

which is a positive measure. From (9) we see that

$$\begin{aligned} 0 &\le \Big(\sqrt{(1-x)(1-y)t} - \sqrt{xyu} \,\Big)^2 \\ &\le \Phi(x,y,t;u,\varphi) \le \Big(\sqrt{(1-x)(1-y)} + \sqrt{xy} \,\Big)^2 \\ &\le \Big(\frac{1-x+1-y}{2} + \frac{x+y}{2} \Big)^2 = 1 \end{aligned}$$

for $0 \le x, y, u, t \le 1$. Whence - by remark (d) above - the sequence in (8) is completely monotonic. This finishes the proof of Lemma 4, and settles the proof of Theorem 2.

Remark. For ultraspherical polynomials, the representation formula of Lemma 5 has the following limit as $\alpha \to \beta = \lambda - \frac{1}{2}$,

$$\frac{C_n^{(\lambda)}(x)}{C_n^{(\lambda)}(1)} = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_0^\pi \Big[x + \sqrt{x^2-1}\, \cos\varphi \Big]^n \left(\sin\varphi\right)^{2\lambda-1} d\varphi\,, \quad \lambda > 0\,.$$

In particular, for the Legendre polynomials P_n normalized by $P_n(1) = 1$ (case $\lambda = \frac{1}{2}$), we recover the Laplace integral

$$P_n(x) = \frac{1}{\pi} \int_0^\pi \Big[x + \sqrt{x^2-1}\, \cos\varphi \Big]^n d\varphi\,.$$

For details, see again [2] and [21].

5. The Quasi-Interpolants

The following second order differential operator $\mathbf{U}_\mu$ plays a prominent role in our analysis,

$$\begin{aligned} -\mathbf{U}_\mu := &\sum_{i=1}^{d} \left(\omega_\mu(\mathbf{x})\right)^{-1} \frac{\partial}{\partial x_i} \Big\{ \omega_\mu(\mathbf{x})\, x_0 x_i \,\frac{\partial}{\partial x_i} \Big\} \\ &+ \sum_{1 \le i < j \le d} \left(\omega_\mu(\mathbf{x})\right)^{-1} \Big(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_i} \Big) \Big\{ \omega_\mu(\mathbf{x})\, x_i x_j \,\Big(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_i} \Big) \Big\}\,. \end{aligned} \tag{10}$$

Here, as before, $\mathbf{x} = (x_0, x_1, \ldots, x_d)$ with $x_0 = 1 - x_1 - \cdots - x_d$. In the definition of the operator, we take the negative sign in order to have a positive spectrum.

Lemma 6. *The differential operator $\mathbf{U}_\mu$ is densely defined on the Hilbert space $\mathbf{H}$, and symmetric. We have*

$$\mathbf{U}_\mu(p_m) = m(m+d+|\mu|)\, p_m\,, \quad m \ge 0\,, \quad p_m \in \mathbf{E}_{m,\mu}\,,$$

i.e., the spaces $\mathbf{E}_{m,\mu}$ are also eigenspaces of $\mathbf{U}_\mu$. In particular,

$$\mathbf{U}_\mu(\mathbf{M}_{n,\mu}(f)) = \mathbf{M}_{n,\mu}(\mathbf{U}_\mu(f))$$

for $f \in C^2(\mathbf{S}^d)$.